A MONOPOLE HOMOLOGY FOR INTEGRAL HOMOLOGY 3-SPHERES

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ABSTRACT. To an integral homology 3-sphere Y, we assign a well-defined $\mathbf{Z}$-graded (monopole) homology $MH_*(Y, I_\eta(\Theta; \eta_0))$ whose construction in principle follows from the instanton Floer theory with the dependence of the spectral flow $I_\eta(\Theta; \eta_0)$, where Θ is the unique $U(1)$-reducible monopole of the Seiberg-Witten equation on Y and η_0 is a reference perturbation datum. The definition uses the moduli space of monopoles on $Y \times \mathbf{R}$ introduced by Seiberg-Witten in studying smooth 4-manifolds. We show that the monopole homology $MH_*(Y, I_\eta(\Theta; \eta_0))$ is invariant among Riemannian metrics with same $I_\eta(\Theta; \eta_0)$. This provides a chamber-like structure for the monopole homology of integral homology 3-spheres. The assigned function $MH_{SWF} : \{I_\eta(\Theta; \eta_0)\} \rightarrow \{MH_*(Y, I_\eta(\Theta; \eta_0))\}$ is a topological invariant (as Seiberg-Witten-Floer Theory).

1. INTRODUCTION

Since Donaldson [7] initiated the study of smooth 4-manifolds via the Yang-Mills theory, the gauge theory (Donaldson invariants, relative Donaldson-Floer invariants and Taubes' Casson-invariant interpretation, etc) has proved remarkably fruitful and rich to unfold some of the mysteries in studying smooth 4-manifolds. The topological quantum field theory proposed by Witten [25] stimulates the most exciting developments in low-dimensional topology. In 1994, Seiberg and Witten introduces a new (simpler) kind of differential-geometric equation (see [6, 26]). In a very short time after the equation was introduced, some long-standing problems were solved, new and unexpected results were discovered. For instance, Kronheimer and Mrowka [12] proved the Thom conjecture affirmatively, several authors proved variants (generalizations) of the Thom conjecture independently in [8, 17, 20], as well as the three-dimensional version of the Thom conjecture [4]. Taubes showed that there are more constraints on symplectic structures in [21, 22] and the beautiful equality $SW = Gr$ in [23, 24]. See [6] for a survey in the Seiberg-Witten theory.

Using the dimension-reduction principle, one expects the Floer-type homology of 3-manifolds via the Seiberg-Witten equation. Indeed Kronheimer and Mrowka [12] analyzed the Seiberg-Witten-Floer theory for $\Sigma \times S^1$, where Σ is a closed oriented surface. Later on Marcolli studied the Seiberg-Witten-Floer homology for 3-manifolds with first Betti number positive in [15]. For a connected compact oriented 3-manifold with positive first Betti number and zero Euler characteristic, Meng and Taubes [16] showed that a (average) version of Seiberg-Witten invariant is the same as the Milnor torsion. The interesting class of 3-manifolds as integral (rational) homology 3-spheres is lack of well-posed theory. Although various authors attempted to resolve the problem on defining a "Seiberg-Witten-Floer" theory, the new phenomenon of harmonic-spinor jumps and the dependence

of Riemannian metrics is not addressed clearly. The metric-dependence (also related to the harmonic-spinors) issue is quickly realized by many experts in this field (see [6, 18]). In [18], the irreducible Seiberg-Witten-Floer homology of Seifert space is shown to be dependent on the metric and the choice of connection on the tangent bundle (as our reference η_0 in this paper).

In this paper, we construct a monopole homology from the Seiberg-Witten equation in the same way as an instanton Floer homology from the Self-Duality equation in Donaldson-Floer theory [9]. Our key point is that by using the unique $U(1)$-reducible solution Θ of the Seiberg-Witten equation on an integral homology 3-sphere Y we make use of the spectral flow of Θ to capture the dependence in certain perturbation classes of Riemannian metrics and 1-forms. The same idea was used before by the present author to establish a symplectic Floer homology of knots in [14], and the original one was in the study of the instanton Floer homology of rational homology 3-spheres by Lee and the present author in [13]. Many technique issues such as transversality, transitivity and gluing property are treated in many authors books and papers, those techniques follow the same line in [9] or simpler. So we omit the details on these, but only emphasize the Riemann-metric dependence and understand the role of such a fixing spectral flow of $(\Theta; \eta_0)$.

Our approach is similar to approaches in [5, 13, 14] to understand the perturbation data (including Riemannian metrics). The unique $U(1)$-reducible Θ gives a spectral flow $I_\eta(\Theta; \eta_0)$ as a Maslov index in [5] Part III. The spectral flow $I_\eta(\Theta; \eta_0) = \mu_\eta(\Theta) - \mu_{\eta_0}(\Theta)$ with respect to a reference η_0 fixes a class of admissible perturbations consisting of Riemannian metrics and 1-forms. As long as Riemannian metrics and 1-forms give the same spectral flow $I_\eta(\Theta; \eta_0)$, we prove that the constructed monopole homology is invariant inside the fixed class of Riemann-metrics and 1-forms ($\eta = (g_Y, \alpha)$) with same $I_\eta(\Theta; \eta_0)$. The spectral flow $I_\eta(\Theta; \eta_0)$ is not a topological invariant, and is dependent upon the Riemannian metrics. Without fixing a class of Riemannian metrics with same $I_\eta(\Theta; \eta_0)$, one cannot obtain well-defined notions such as spectral flow of irreducible Seiberg-Witten solutions on Y, and the gluing formula as well as the relative Seiberg-Witten invariant. Hence our results follow from fixing $I_\eta(\Theta; \eta_0)$.

Theorem A. *(1) For an integral homology 3-sphere Y and any admissible perturbation η, there is a well-defined $\mathbf{Z}$-graded monopole homology $MH_*(Y, I_\eta(\Theta; \eta_0))$ constructed by the Seiberg-Witten equation over $Y \times \mathbf{R}$.*

(2) For any two admissible perturbations η_1 and η_2, there is a group homomorphism Ψ_* between two monopole homologies $MH_*(Y, I_{\eta_1}(\Theta; \eta_0))$ and $MH_*(Y, I_{\eta_2}(\Theta; \eta_0))$.

(3) If $I_{\eta_1}(\Theta; \eta_0) = I_{\eta_2}(\Theta; \eta_0)$, then the homomorphism Ψ_* is an isomorphism.

Our fixed-class $I_\eta(\Theta; \eta_0)$ of Riemannian metrics gains control of the birth and death of irreducible solutions of the Seiberg-Witten equation on the integral homology 3-sphere Y. Changing the reference η_0 into $\eta_0^{'}$ corresponds to the overall degree-shifting by $\mu_{\eta_0^{'}}(\Theta) - \mu_{\eta_0}(\Theta)$ for monopole homologies. The control in the instanton homology of rational homology 3-spheres is gained by fixing the spectral flows of all $U(1)$-reducibles from the Wilson-loop perturbations (not metrics). The control in the monopole homology of integral homology 3-spheres is gained by fixing the spectral flow

of the unique $U(1)$-reducible Θ from the Riemannian metrics (not only 1-forms). Fixing $I_\eta(\Theta; \eta_0)$ enters crucially in proving Theorem A and Theorem B.

Theorem B. *For a smooth 4-manifold $X = X_0 \#_Y X_1$ with $b_2^+(X_i) > 0 (i = 0, 1)$ and Y an integral homology 3-sphere, the Seiberg-Witten invariant of X is given by the Kronecker pairing of $MH_*(Y; I_\eta(\Theta; \eta_0))$ with $MH_{-1-*}(-Y; I_\eta(\Theta; \eta_0))$ for the relative Seiberg-Witten invariants $q_{X_0,Y,\eta}$ and $q_{X_1,-Y,\eta}$ (see Definition 8);*

$$\langle, \rangle : MH_*(Y; I_\eta(\Theta; \eta_0)) \times MH_{-1-*}(-Y; I_\eta(\Theta; \eta_0)) \to \mathbf{Z}; \quad q_{SW}(X) = \langle q_{X_0,Y,\eta}, q_{X_1,-Y,\eta} \rangle.$$

The paper is organized as follows. §2 provides an introduction of the Seiberg-Witten equation on 3-manifolds. §3 studies the configuration space over Y through Seiberg-Witten equation and a natural monopole complex. We show that there are admissible perturbations from Riemannian metrics and 1-forms in §4 via the method similar to [19]. The spectral-flow properties and dependence on Riemannian metrics are discussed in §5. The proof of Theorem A (Proposition 6.4 for (1), Proposition 7.1 for (2) and Proposition 7.2 for (3)) is occupied in §6 and §7. In §8, we study the relative Seiberg-Witten invariant and complete the proof of Theorem B as Theorem 8.4.

2. Seiberg-Witten equation on 3-manifolds

It is well-known that every closed oriented 3-manifold is spin. The group $Spin(3) \cong SU(2) \cong Sp_1$ is the universal covering of $SO(3) = Spin(3)/\{\pm I\}$. Pick a Riemannian metric g on Y. The metric g defines the principal $SO(3)$-bundle $P_{SO}(Y)$ of oriented orthonormal frames on Y. A spin structure is a lift of $P_{SO}(Y)$ to a principal $Spin(3)$-bundle $P_{Spin}(Y)$ over Y. The set of equivalence classes of such lifts has, in a natural way, the structure of a principal $H^1(Y, Z_2)$-bundle over a point. So there is a unique spin-structure on the integral homology 3-sphere Y.

There is a natural adjoint representation

$$Ad : Spin(3) \times Sp_1 \to Sp_1; \quad (q, \alpha) \mapsto q\alpha q^{-1},$$

and associated rank-2 complex vector bundle (spinor bundle)

$$W = P_{Spin(3)}(Y) \times_{Ad} C^2.$$

Let $L = detW$ be the determinant line bundle. For the ordinary Spin-structure, one has a Clifford multiplication

$$c : T^*Y \otimes W \to W$$

$$c([p, \alpha]) \otimes [p, v] \to [p, \overline{\alpha}v].$$

So c induces a map $T^*Y \to Hom(W, W)$. The spinor pairing $\tau : W \otimes \overline{W} \to T^*Y$ is given by

$$[p, v_1 \otimes v_2] \to \tau(\frac{1}{4}Im(v_1 iv_2)),$$

where τ is an orientation preserving isomorphism $P_{Spin(3)}(Y) \times Sp_1 \to T^*Y$. A connection a on L together with the Levi-Civita connection on the tangent bundle of Y form a covariant derivative on

W. This maps sections of W into sections of $W \otimes T^*Y$. Followed by the Clifford multiplication, one has a Dirac operator

$$\partial_a^g : \Gamma(W) \xrightarrow{\nabla_a^g} \Gamma(W \otimes T^*Y) \xrightarrow{c} \Gamma(W).$$

The determinant line bundle L is trivial for the spin structure, so we may choose θ to be the trivial connection and $\partial_\theta^g : \Gamma(W) \to \Gamma(W)$ is the usual Dirac operator. Note that all bundles over the integral homology 3-sphere Y are **trivial**.

There is a unique spin-structure on $Y \times \mathbf{R}$ associated to the unique spin-structure on Y with the product metric on $Y \times \mathbf{R}$. The two spinor bundles $W^{\pm}$ on $Y \times \mathbf{R}$ can be identified by using a Clifford multiplication by dt, where t is denoted for the variable on $\mathbf{R}$. Both W^+ and W^- are obtained by the pull-back of the $U(2)$-bundle $W \to Y$ from the projection map $Y \times \mathbf{R} \to Y$. Thus we have the identification of the map $\sigma : \Lambda^2 T^*(Y \times \mathbf{R}) \to Hom(W^+, W^-)$ and the map $\tau^{-1} : T^*Y \to Hom(W, W)$ through the above identifications. $\sigma(\eta) = \tau^{-1}(*_g \eta)$. In other words from the identification $\Lambda^2 T^*(Y \times \mathbf{R}) = \Lambda^2 T^*Y \oplus \Lambda^1 T^*Y$ and using the Hermitian pairing on $W^{\pm}$, there is an induced pairing

$$\tau : \overline{W} \times W \to \Lambda^1 T^*Y.$$

In fact for every $\gamma : T^*Y \to Hom(W, W)$ (a spin structure), that is a way to determine a spin structure on $Y \times \mathbf{R}$ by

$$\sigma : T^*(Y \times \mathbf{R}) \to Hom(W \oplus W, W \oplus W); \quad \sigma(v, r) = \begin{pmatrix} 0 & \gamma(v) + r1 \\ \gamma(v) - r1 & 0 \end{pmatrix}.$$

The determinant line bundle $L_{(4)} = det W^{\pm}|_{Y \times \mathbf{R}}$ (a trivial line bundle) carries $U(1)$-connections $A = a + \phi dt$. So the Dirac operator D_A^g for the product metric $g + dt^2$ over $Y \times \mathbf{R}$ is given by

$$D_A^g = \begin{pmatrix} 0 & -\nabla_t + \partial_a^g \\ \nabla_t + \partial_a^g & 0 \end{pmatrix},$$

where ∂_a^g is a twisted self-adjoint Dirac operator on $\Gamma(W) \to \Gamma(W)$, and $\nabla_t = \frac{\partial}{\partial t} + \phi$ is a twisted skew adjoint Dirac operator over $\mathbf{R}$.

The curvature 2-form of $A = a + \phi dt$ can be calculated as $F_A = F_a + (\frac{\partial a}{\partial t} - d_a \phi) dt$. Using the identification of $\Omega^2(Y \times \mathbf{R}) \cong \Omega^2(Y) \oplus \Omega^1(Y)$, we can write F_A^+ as $*_g F_a + (\frac{\partial a}{\partial t} - d_a \phi) \in \Omega^1(Y)$ as the self-dual component of the curvature F_A. Now the Seiberg-Witten monopole equation on 4-manifolds reduces to a Seiberg-Witten monopole equation on 3-manifolds as

$$\begin{cases} (\nabla_t + \partial_a^g)\psi & = 0 \\ *_g F_a + (\frac{\partial a}{\partial t} - d_a \phi) & = i\tau(\psi, \psi) \end{cases} \tag{2.1}$$

for $\psi \in \Gamma(W)$. It is equivalent to the flow equation of (a, ϕ, ψ):

$$\begin{cases} \frac{\partial \psi}{\partial t} & = -\partial_a^g \psi - \phi.\psi \\ \frac{\partial a}{\partial t} & = - *_g F_a + d_a \phi + i\tau(\psi, \psi). \end{cases} \tag{2.2}$$

The equation (2.1) is invariant under the gauge transformation $u \in Map(Y, U(1))$, where the gauge group action on $(a + \phi dt, \psi)$ is given by

$$u \cdot (a + \phi dt, \psi) = (u^* a + (\phi - u^{-1} \frac{du}{dt}) dt, \psi u^{-1}). \tag{2.3}$$

There is a temporal gauge to obtain a simpler equation. The temporal gauge u is the element which $u \cdot (a + \phi dt) = u^*a$, i.e., $\phi - u^{-1}\frac{du}{dt} = 0$. Then the equation (2.2) can be reduced to the following form.

$$\begin{cases} \frac{\partial \psi}{\partial t} & = -\partial_a^g \psi \\ \frac{\partial a}{\partial t} & = - *_g F_a + i\tau(\psi, \psi). \end{cases} \tag{2.4}$$

3. Configuration spaces on Y

Fix a trivialization $L = Y \times U(1)$, one can identify the space of $U(1)$-connections of Sobolev L_k^p-norm with the space $\mathcal{A}_k^p = L_k^p(\Omega^1(Y, i\mathbf{R}))$ of 1-forms on Y such that the zero element in $\Omega^1(Y, i\mathbf{R})$ corresponds to the trivial connection θ on L. The gauge group of L can be identified with $\mathcal{G}_k^p(Y) = L_{k+1}^p(Map(Y, U(1)))$ acting on $\mathcal{A}_k^p \times L_k^p(\Gamma(W))$ by (2.3). We need to assume that $k + 1 > 3/p$ so that $\mathcal{G}_Y = \mathcal{G}_k^p(Y)$ is a Lie group. We may take $k = 1, p = 2$.

Let $\mathcal{C}_Y$ be the configuration space

$$\mathcal{C}_Y = L_k^2(\{\Omega^1 \oplus \Omega^0\}(Y, i\mathbf{R}) \oplus \Gamma(W)).$$

The quotient space is $\mathcal{B}_Y = \mathcal{C}_Y/\mathcal{G}_Y$. Denote $\mathcal{C}_Y^* = \{(a, \phi, \psi) \in \mathcal{C}_Y | \psi \neq 0\}$. For $(a, \phi, \psi) \in \mathcal{C}_Y^*$, the isotropy group $\Gamma_{(a,\phi,\psi)} = \{id\}$. For $(a, \phi, \psi) \in \mathcal{C}_Y \setminus \mathcal{C}_Y^*$, the isotropy group $\Gamma_{(a,\phi,0)} = U(1)$, these elements are called reducibles. For example, $\Theta = (\theta, 0, 0)$ is reducible by all constant maps from Y to $U(1)$. Note that $\mathcal{G}_Y$ acts freely on $\mathcal{C}_Y^*$, so $\mathcal{B}_Y^* = \mathcal{C}_Y^*/\mathcal{G}_Y$ forms an open and dense set in $\mathcal{C}_Y/\mathcal{G}_Y$.

Proposition 3.1. *$\mathcal{B}_Y^*$ is a Hilbert manifold. For $(a_0, \phi_0, \psi_0) \in \mathcal{C}_Y^*$, the tangent space of $\mathcal{B}_Y^*$ can be identified with*

$$T_{[(a_0,\phi_0,\psi_0)]}\mathcal{B}_Y^* = \{(a, \phi, \psi) \in L_k^2(\{\Omega^1 \oplus \Omega^0\}(Y, i\mathbf{R}) \oplus \Gamma(W))|$$
$$\|(a, \phi, \psi)\|_{L_{k-1}^2} < \varepsilon, \quad d_{a_0}^* \psi + Im(\psi_0, \psi) = 0\}.$$

Proof: This follows from the construction of slice in [7, 10]. It will be clear from context to identify (a_0, ϕ_0, ψ_0) with its gauge equivalence class in our notation. The gauge orbit of $(a_0, \phi_0, \psi_0) \in \mathcal{C}_Y^*$ is given by $\mathcal{G}_Y \to \mathcal{C}_Y^*$:

$$g = e^{iu} \to (a_0 - g^{-1}dg, \phi_0, \psi_0 g^{-1}).$$

The linearization of this map at $Id = e^0$ is

$$\delta_0 : T_{id}\mathcal{G}_Y = \Omega^0(Y, i\mathbf{R}) \to \{\Omega^1 \oplus \Omega^0\}(Y, i\mathbf{R}) \oplus \Gamma(W)$$

$$u \mapsto (-du, 0, -\psi_0 u).$$

So the adjoint operator δ_0^* of δ_0 is given by

$$\delta_0^* \psi = d_{a_0}^* \psi + Im(\psi_0.\psi).$$

A neighborhood of $[(a_0, \phi_0, \psi_0)] \in \mathcal{B}_Y^*$ can be described as a quotient of $T_{[(a_0,\phi_0,\psi_0)],\varepsilon}\mathcal{B}_Y^*/\Gamma_{(a_0,\phi_0,\psi_0)}$ for sufficiently small ε. Every nearby orbit meets the slice $(a_0, \phi_0, \psi_0) + T_{[(a_0,\phi_0,\psi_0)],\varepsilon}\mathcal{B}_Y^*$. This is amount to solving the gauge fixing condition relative to (a_0, ϕ_0, ψ_0), i.e., there exists a unique $u \in \Omega^0(Y, i\mathbf{R})$ such that $e^{iu} \cdot (a_0 + a, \phi_0 + \phi, \psi_0 + \psi) \in T_{[(a_0,\phi_0,\psi_0)],\varepsilon}\mathcal{B}_Y^*$ for $\psi_0 \neq 0$. Hence it follows from applying the implicit function theorem. $\qquad\square$

There is an associated bundle $\mathcal{C}_Y^* \times_{\mathcal{G}_Y} (\Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W))$ over $\mathcal{C}_Y^*$ because of the free action of $\mathcal{G}_Y$ on $\mathcal{C}_Y^*$. We define a section $f : \mathcal{C}_Y^* \to \mathcal{C}_Y^* \times_{\mathcal{G}_Y} (\Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W))$ by

$$f(a, \phi, \psi) = [(a, \phi, \psi), *_g F_a - d_a \phi - i\tau(\psi, \psi), \partial_a^g \psi + \phi.\psi].$$

Note that f is $\mathcal{G}_Y$-equivariant, $f(g \cdot (a, \phi, \psi)) = g \cdot f(a, \phi, \psi)$. Hence it descends to $\mathcal{B}_Y^*$,

$$f : \mathcal{B}_Y^* \to \mathcal{C}_Y^* \times_{\mathcal{G}_Y} (\Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W)).$$

Now $f(a, \phi, \psi) \in T_{[(a,\phi,\psi)],\varepsilon} L_{k-1}^2 \mathcal{B}_Y^* = \mathcal{L}_{[(a,\phi,\psi)]}$. So f can be thought of as a vector field on the Hilbert manifold $\mathcal{B}_Y^*$. Over $\mathcal{B}_Y^*$, f is a section of the bundle $\mathcal{L}$ with fiber $\mathcal{L}_{[(a,\phi,\psi)]}$.

Definition 3.2. The zero set of f in $\mathcal{B}_Y^*$ is the moduli space of solutions of the 3-dimensional Seiberg-Witten equation

$$f^{-1}(0) = \mathcal{R}_{SW}^*(Y, g) = \{[(a, \phi, \psi)] \in \mathcal{C}_Y^* \text{ satisfies } (3.1)\}/\mathcal{G}_Y.$$

$$\begin{cases} \partial_a^g \psi + \phi.\psi = 0 \\ *_g F_a - d_a \phi - i\tau(\psi, \psi) = 0 \end{cases}. \tag{3.1}$$

We will show that $\mathcal{R}_{SW}^*(Y, g)$ is a zero-dimensional smooth manifold and its algebraic number is the Euler characteristic of a monopole homology defined in §6 (see also [4] for instance).

The linearization of f can be computed as the following.

$$\begin{aligned} f(a_0 + sa, \phi_0 + s\phi, \psi_0 + s\psi) &= (*_g F_{a_0+sa} - d_{a_0+sa}(\phi_0 + s\phi) - i\tau(\psi_0 + s\psi, \\ &\quad \psi_0 + s\psi), \partial_{a_0+sa}^g(\psi_0 + s\psi) + (\phi_0 + s\phi).(\psi_0 + s\psi) \\ &= f(a_0, \phi_0, \psi_0) + s\delta_1(a_0, \phi_0, \psi_0)((a, \phi, \psi)) + o(s^2). \end{aligned}$$

So the linearized operator $Df(a_0, \phi_0, \psi_0) = \delta_1(a_0, \phi_0, \psi_0) : T_{[(a_0,\phi_0,\psi_0)]}\mathcal{B}_Y^* \to \mathcal{L}_{[(a_0,\phi_0,\psi_0)]}$ is given by

$$\delta_1(a_0, \phi_0, \psi_0) : \{\Omega^1 \oplus \Omega^0\}(Y, i\mathbf{R}) \oplus \Gamma(W) \to \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W),$$

$$((a, \phi, \psi) \longmapsto \begin{pmatrix} *_g d_{a_0} & -d_{a_0} & -iIm(\psi_0, , \cdot) \\ c(\cdot\psi_0) & c \cdot \psi_0 & \partial_{a_0}^g + \phi_0\cdot \end{pmatrix} \begin{pmatrix} a \\ \phi \\ \psi \end{pmatrix}.$$

It forms a natural 3-dimensional monopole complex, since $\ker \delta_0^*$ is the gauge fixing slice. So

$$MC_\bullet : 0 \to \Omega^0(Y, i\mathbf{R}) \xrightarrow{\delta_0} \{\Omega^1 \oplus \Omega^0\}(Y, i\mathbf{R}) \oplus \Gamma(W) \xrightarrow{\delta_1} \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W) \to 0, \tag{3.2}$$

is a short exact sequence. The operator

$$\delta_0^* \oplus \delta_1(a_0, \phi_0, \psi_0) : \{\Omega^1 \oplus \Omega^0\}(Y, i\mathbf{R}) \oplus \Gamma(W) \to \{\Omega^1 \oplus \Omega^0\}(Y, i\mathbf{R}) \oplus \Gamma(W)$$

$$(a, \phi, \psi) \longmapsto \begin{pmatrix} *_g d_{a_0} & -d_{a_0} & -iIm(\psi_0, \cdot) \\ -d_{a_0}^* & 0 & Im(\psi_0, \cdot) \\ c(\cdot\psi_0) & c \cdot \psi_0 & \partial_{a_0}^g + \phi_0\cdot \end{pmatrix} \begin{pmatrix} a \\ \phi \\ \psi \end{pmatrix}, \tag{3.3}$$

is a first-order operator with symbol $\sigma(\delta_0^* \oplus \delta_1) = \sigma(\delta)$, where

$$\delta = \begin{pmatrix} *_g d_{a_0} & -d_{a_0} & 0 \\ -d_{a_0}^* & 0 & 0 \\ 0 & 0 & \partial_{a_0}^g \end{pmatrix}$$

is a first-order self-adjoint Dirac operator. Hence

$$
\begin{aligned}
Ind(\delta_0^* \oplus \delta_1) &= Ind(\delta) \\
&= Ind \begin{pmatrix} *_g d_{a_0} & -d_{a_0} \\ -d_{a_0}^* & 0 \end{pmatrix} + Ind \partial_{a_0}^g \\
&= 0.
\end{aligned}
\tag{3.4}
$$

Since the operator $\begin{pmatrix} *_g d_{a_0} & -d_{a_0} \\ -d_{a_0}^* & 0 \end{pmatrix}$ is self-adjoint and every Dirac operator has index zero over odd (3-)dimensional manifolds, thus we have the zero index for the operator $\delta_0^* \oplus \delta_1$. Generically, the moduli space $\mathcal{R}_{SW}(Y, g)$ is zero-dimensional.

Define $H^0(MC_\bullet) = \ker \delta_0$, $H^1(MC_\bullet) = \ker \delta_1 / im \delta_0$, $H^2(MC_\bullet) = coker \delta_1$. The first cohomology $H^1(MC_\bullet)$ is isomorphic for every $(a_0, \phi_0, \psi_0) \in \mathcal{B}_Y^*$, so that $(a_0, \phi_0, \psi_0) \in \mathcal{B}_Y^*$ is a nondegenerate zero of f if and only if $\ker(\delta_0^* \oplus \delta_1) = H^1(MC_\bullet) = 0$. For $\Theta = (\theta, 0, 0)$ and a generic metric g without harmonic spinors of ∂_θ^g, we have that Θ is always isolated and nondegenerate (in the Bott sense) zero of f on the integral homology 3-sphere Y.

4. Admissible Perturbation and Transversality

In this section, we prove that there are enough perturbations to make the zero set of f transverse. There is a 1-form perturbation reduced from 4-dimensional Seiberg-Witten equation as in [6, 12, 21]. In our 3-dimensional case, *the harmonic spinor may vary or jump as metrics on Y vary.* In order to obtain any topological information, one needs to extend the perturbation-data and understand the harmonic spinors accordingly. The method we used here is essentially the one used in [10, 13, 14, 19].

Let $\mathcal{P}_Y = \Sigma_Y \times \Omega^1(Y, i\mathbf{R})$ be the space of perturbation data, where Σ_Y is the space of Riemannian metrics on Y. Consider the union $\cup_{(g,,\alpha) \in \mathcal{P}_Y} \mathcal{R}_{SW}^*(Y; g, \alpha)$ of the moduli spaces of 3-dimensional Seiberg-Witten solutions over all metrics and 1-forms. If the union is a (Banach) Hilbert manifold, then its projection to the space $\mathcal{P}_Y$ is a Fredholm map. So there exists a Baire first category in $\mathcal{P}_Y$ such that $\mathcal{R}_{SW}^*(Y; g, \alpha)$ is a manifold by the Sard-Smale theorem.

Let f_η be the parametrized smooth section of the bundle $\mathcal{L} \to \mathcal{B}_Y^* \times \mathcal{P}_Y$ with $\eta = (g, \alpha) \in \mathcal{P}_Y$. The map f_η is given by

$$
f_\eta : \mathcal{B}_Y^* \to \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W)
$$

$$
(a, \phi, \psi) \mapsto (*_g F_a - d_a \phi - i\tau(\psi, \psi) + \alpha, \partial_a^{\nabla_0 + \alpha} \psi + \phi.\psi),
$$

where ∇_0 is the Levi-Civita connection for the metric g. Let $f_{1\eta}(a, \phi, \psi) = \partial_a^{\nabla_0 + \alpha} \psi + \phi.\psi$ be the second component of the map f_η on $\Gamma(W)$, and $f_{0\eta}(a, \phi, \psi)$ be the first component of f_η.

Lemma 4.1. $f_{1\eta}$ *is a submersion ($Df_{1\eta}$ is surjective).*

Proof: The differential $Df_{1\eta}$ is given by the formula

$$
Df_{1\eta}(a, \phi, \psi; o, \alpha)(\varepsilon a, \varepsilon \phi, \varepsilon \psi, 0, \varepsilon \alpha) = \partial_a^{\nabla_0 + \alpha}(\varepsilon \psi) + (\varepsilon \alpha + \varepsilon a + \varepsilon \phi).\psi + \phi.\varepsilon \psi,
$$

where we vary along the subspace $\{\Omega^1 \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W)\} \times \{\{0\} \times \Omega^1(Y, i\mathbf{R})\}$ of $T^*_{[a,\phi,\psi]}\mathcal{B}^*_Y \times \mathcal{P}_Y$. We want to show that $Df_{1\eta}$ is surjective. Suppose the contrary. Then there exists a spinor $\chi \in \Gamma(W)$ such that it is perpendicular to $ImDf_{1\eta}$.

$$\langle \partial_a^{\nabla_0 + \alpha}(\varepsilon\psi), \chi \rangle = 0, \tag{4.1}$$

for all $\varepsilon\psi$. I.e., $\chi \in \ker(\partial_a^{\nabla_0+\alpha})^*$. By the elliptic regularity of (4.1), a solution χ is smooth. Choose a point $y \in Y$ such that $\chi(y) \neq 0$. By the uniqueness of continuation of the solution of the elliptic equation [2], $\partial_a^{\nabla_0+\alpha} \cdot (\partial_a^{\nabla_0+\alpha})^*\chi = 0$, there is a neighborhood U_y of y such that $\chi(y) \neq 0$ for $y \in U_y$. Thus we can find a 1-form $\varepsilon\alpha + \varepsilon a \in \Omega^1(Y, i\mathbf{R})$ such that $(\varepsilon\alpha + \varepsilon a).\psi = \lambda\chi$ with $\lambda \neq 0$ in U_y, and $\varepsilon\alpha + \varepsilon a$ has compact support. So we obtain

$$\begin{aligned}
0 &= \langle \partial_{a+\varepsilon a}^{\nabla_0+\alpha+\varepsilon\alpha}(\varepsilon\psi), \chi \rangle \\
&= \langle \partial_a^{\nabla_0+\alpha}(\varepsilon\psi), \chi \rangle + \langle (\varepsilon\alpha + \varepsilon a).\varepsilon\psi, \chi \rangle \\
&= \langle \lambda\chi, \chi \rangle = \lambda\langle \chi, \chi \rangle.
\end{aligned}$$

Therefore $\chi = 0$ in U_y, so $\chi \equiv 0$ by a result in [2]. $\square$

By the Hodge decomposition of $\Omega^1(Y, i\mathbf{R}) = Imd \oplus Imd^*$ for Y, we have that δ_1 is surjective. Thus $f_{0\eta}(\alpha, \phi, \psi) = *_g F_a - d_a\phi - i\tau(\psi, \psi) + \alpha$ is also a submersion onto $\Omega^1(Y, i\mathbf{R})$.

Corollary 4.2. *The spaces $f_{0\eta}^{-1}(0)$ and $f_{1\eta}^{-1}(0)$ are Banach manifolds.*

$\square$

Now at point $(a_0, \phi_0, \psi_0; g_0, \alpha) \in \mathcal{C}_Y \times \mathcal{P}_Y$, the parametrized smooth section

$$f(a_0, \phi_0, \psi_0; g_0, \alpha) = f_{(g_0,\alpha)}(a_0, \phi_0, \psi_0) = f_\eta(a_0, \phi_0, \psi_0)$$

is submersion.

Proposition 4.3. *The differential Df is onto at all points of the moduli space $f^{-1}(0) \subset \mathcal{C}^*_Y \times \mathcal{P}_Y$.*

Proof: The differential Df at $(a_0, \phi_0, \psi_0; g_0, \alpha) \in \mathcal{C}_Y \times \mathcal{P}_Y$ is of the form (Df_0, Df_1)

$$\begin{aligned}
Df_0 &= *_{g_0}d_{a_0}a + (g)_*F_{a_0} - d_{a_0}\phi - iIm(\psi_0, \psi) - a.\phi_0 + \alpha \\
Df_1 &= \partial_{a_0}^{\nabla_0+\alpha_0}\psi + (\alpha + a).\psi_0 + (\phi.\psi_0 + \phi_0.\psi) + r(g))
\end{aligned}$$

where $(g)_*$ is the variation of the Hodge star operator $(g)_* = \frac{d}{ds}|_{s=0}*_{g_0+sg}$, $r(g)$ is a zero order operator applied to the variation $g_0 + sg + o(s^2)$ of metric, $a.\phi_0$ is the Clifford multiplication of 1-form a on the section $\phi_0 \in \Gamma(W)$. The surjective of Df_0 follows from Theorem 3.1 of [10], and the surjective of Df_1 follows from Proposition I.3.5 of [19] (see also [4, 15, 16, 18]). $\square$

We consider the map $f_* : \mathcal{C}^*_Y \times \mathcal{P}_Y \to \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W)$.

Corollary 4.4. *The space $f_*^{-1}(0)$ is a Banach manifold.*

Proof: Take f_* as a section of $\mathcal{B}_Y^* \times \mathcal{P}_Y$ to $(\mathcal{C}_Y^* \times_{\mathcal{G}_Y} (\Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W)) \times \mathcal{P}_Y$. So $f_*^{-1}(0)|_{\mathcal{B}_Y^*} = f_*^{-1}(0)/\mathcal{G}_Y$ is a Banach manifold.

$$\begin{array}{ccc} \mathcal{C}_Y^* \times \mathcal{P}_Y & \xrightarrow{f} & \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W) \\ \downarrow \pi_2 & & \\ \mathcal{P}_Y & & \end{array}$$

The projection map π_2 is a smooth Fredholm map of index zero. It follows exactly from the same argument in [7, 10]. $\qquad\square$

Corollary 4.5. *The inverse image* $\pi_2^{-1}((g, \alpha))$ *of a generic parameter* $(g, \alpha) \in \mathcal{P}_Y$, *the moduli space* $\mathcal{R}_{SW}(Y, (g, \alpha))$ *of the 3-dimensional monopole solutions is a zero dimensional manifold.*

A perturbation $\eta = (g, \alpha)$ satisfying Corollary 4.5 is called **admissible.** In general, the class of reducible elements in $\mathcal{C}_Y \setminus \mathcal{C}_Y^*$ forms a singular strata in the quotient space $\mathcal{B}_Y$. If it is a solution of 3-dimensional Seiberg-Witten equation, it is also singular to the space of $\mathcal{R}_{SW}(Y, g)$. The reducible solutions of the 3-dimensional Seiberg-Witten equation satisfy

$$\begin{aligned} \partial_a^{\nabla_0 + \alpha}\psi + \phi_0.\psi &= 0 \\ -*F_a + d_a\phi &= 0, \end{aligned} \qquad (4.2)$$

for $\psi = 0$. Applying the temporal gauge $g \cdot (a, \phi) = (g^*a, 0)$, we get that g^*a is a flat connection on $Y \times U(1)$ over Y. For integral homology 3-sphere, there is a unique $U(1)$ reducible connection, namely the trivial one. So the reducible solution is $(\theta, 0)$. There is a unique $U(1)$-reducible solution of (4.2), denoted by $\Theta = (\theta, 0)$.

Note that $\ker \delta_1 = \ker \partial_a^g$ for an integral homology 3-sphere. For a generic metric g, $\ker \partial_a^g = 0$. But $\ker \partial_a^{g_t}$ may have a nontrivial kernel as the Riemannian metrics vary in an one-parameter family (see [11]). The harmonic spinor, even the dimension of the harmonic spinor, depends on the metric used in defining the Dirac operator. Hence the harmonic-spinor jump creates and/or destroys irreducible solutions of the 3-dimensional Seiberg-Witten equation. This is the main problem to understand the new phenomenon that the "Seiberg-Witten-Floer theory" is not entirely metric-independent (see [6]). In the next section, we study such a dependence of Riemannian metrics.

Proposition 4.6. $\mathcal{R}_{SW}^*(Y, (g, \alpha)) = \mathcal{R}_{SW}(Y, (g, \alpha)) \setminus \{\Theta\}$ *is a zero-dimensional smooth manifold for a first category near* (g, α) *in* $\mathcal{P}_Y$.

Proof: The results follows from the construction above, Proposition 2c.1 of [9] and the Sard-Smale theorem. $\qquad\square$

Define the weighted Sobolev space $L_{k,\delta}^p$ on sections ξ of a bundle over $Y \times \mathbf{R}$ to be the space of ξ for which $e_\delta \cdot \xi$ is in L_k^p, where $e_\delta(y, t) = e^{\delta|t|}$ for $|t| \geq 1$. For any $\delta \geq 0$ and any Seiberg-Witten monopole solution (A, Φ) on $Y \times \mathbf{R}$, the linearized operator

$$D_{A,\Phi} : L_{k+1,\delta}^p(\Gamma(W_{(4)}^+) \oplus \Omega^1(Y \times \mathbf{R})) \to L_{k,\delta}^p(\Gamma(W_{(4)}^-) \oplus (\Omega^0 \oplus \Omega_+^2)(Y \times \mathbf{R}))$$

is Fredholm (see [6, 9, 12, 21, 26]). We call (A, Φ) *regular* if $\mathrm{Coker} D_{A,\Phi} = 0$ and we call $\mathcal{M}_{Y \times \mathbf{R}}$ (the moduli space of perturbed Seiberg-Witten solutions with finite energy) *regular* if it contains orbits of regular (A, Φ)'s.

Proposition 4.7. *The finite energy condition forces elements of $\mathcal{M}_{Y \times \mathbf{R}}$ to converge to zeros of $f_\eta^{-1}(0)$ on the ends of $Y \times \mathbf{R}$. The set of all perturbations $\eta \in \mathcal{P}_Y$ of which $\mathcal{M}_{Y \times \mathbf{R}}$ is regular is of Baire's first category.*

Proof: The proof follows exactly from the same method in [9] Proposition 2c.2 with Chern-Simons Seiberg-Witten functional as defined in [12] §4 and [4, 15, 16]. $\square$

5. SPECTRAL FLOW AND DEPENDENCE ON RIEMANNIAN METRICS

In this section, we use the unique $U(1)$-reducible solution Θ to capture the metric-dependent relation via the spectral flow. In [13] joined with Lee, the author used the Walker correction-term around $U(1)$-reducibles to obtain homotopy classes of admissible perturbations (realized by a family of Lagrangians), and to show the invariance among the same homotopy class of the Lagrangian perturbations. Those Walker correction-term can be interpreted as the spectral flow in [5, 13].

Proposition 5.1. *For an admissible perturbation $\eta = (g, \alpha) \in \mathcal{P}_Y$ and a nondegenerate zero $(a, \phi, \psi) \in \mathcal{R}_{SW}(Y, \eta) = f_\eta^{-1}(0)$, we can associate an integer $\mu_\eta(a, \phi, \psi) \in \mathbf{Z}$ such that for $(A, \Phi) \in \mathcal{B}_{Y \times \mathbf{R}}((a, \phi, \psi), (a', \phi', \psi'))$*

$$\mu_\eta(e^{iu} \cdot (a, \phi, \psi)) = \mu_\eta(a, \phi, \psi),$$
$$\mathrm{Index} D_{A,\Phi} = \mu_\eta(a, \phi, \psi) - \mu_\eta(a', \phi', \psi') - dim\Gamma_{(a', \phi', \psi')},$$

where $\Gamma_{(a', \phi', \psi')}$ is the isotropy subgroup of (a', ϕ', ψ').

Proof: Let $\pi_1 : Y \times [0, 1] \to Y$ be the projection on the first factor. Let $L_{(4)} \times W_{(4)}$ be the pullback $\pi_1^*(det W^\pm) \times \pi_1^* W^\pm$ such that $(A, \Phi) \in \mathcal{A}_{L_{(4)}} \times W_{(4)}$ satisfies $(A, \Phi)|_{t \leq 0} = (a, \phi, \psi)$ and $(A, \Phi)|_{t \geq 1} = (a', \phi', \psi')$. We have $D_{A,\Phi} = \frac{\partial}{\partial t} + \delta_t$ with $\delta_t = \delta_{A(t), \Phi(t)}$ in (3.3). Then the Fredholm index of $D_{A,\Phi}$ is given by the spectral flow of δ_t (see [3, 5, 9]). The second equality follows from the same proof of Proposition 2b. 2 in [9]. The first equality follows from

$$SF(e^{iu} \cdot (a, \phi, \psi), (a, \phi, \psi)) = \mathrm{Ind} D_{A,\Phi}((a, \phi, \psi), (a, \phi, \psi))_{Y \times S^1}$$
$$= \frac{1}{4}(c_1(L_{(4)})^2 - (2\chi + 3\sigma))(Y \times S^1) = 0,$$

where χ and σ are the Euler number and signature of $Y \times S^1$, and $c_1(L_{(4)})^2(Y \times S^1) = 0$ for the integral homology 3-sphere Y. $\square$

Note that the relative index is gauge-invariant, but depending on the perturbation $\eta \in \mathcal{P}_Y$ by Proposition 5.1. The absolute index may not be well-defined since $\mu_\eta(\Theta)$ depends upon $\eta \in \mathcal{P}_Y$. In the instanton case, we fix the trivialization of a principal bundle and a fixed tangent vector to the trivial connection to determine $\mu(\theta) = 0$ for the trivial connection θ. It turns out that such a

fixation is independent of metrics and other perturbation data in the instanton Floer theory. But this is no longer true for the monopole case.

Proposition 5.2. (Definition) *Two admissible perturbations η_0 and η_1 in $\mathcal{P}_Y$ are (called) homotopic to each other through a 1-parameter family $\eta_t (0 \leq t \leq 1)$ in $\mathcal{P}_Y$ if and only if $\mu_{\eta_0}(\Theta) = \mu_{\eta_1}(\Theta)$.*

Proof: For two admissible perturbations η_0 and η_1 in §4, we can connect them into a 1-parameter family η_t such that there are at most finitely many $t \in (0,1)$ with η_t corresponding harmonic-spinor jumps. Denote those $0 < t_0 < t_1 \cdots < t_n < 1$ and $\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1} = 0$ so that λ_i is not the eigenvalues of $\delta_t = \delta_t(\theta, 0)$ for $t_{i-1} \leq t \leq t_i$, where $t_{-1} = 0$ and $t_{n+1} = 1$. Define $n_i = \dim(\delta_{t_i} - \lambda Id)$ with $\lambda \in [\lambda_{i+1}, \lambda_i]$ and $n_i = -\dim(\delta_{t_i} - \lambda Id)$ with $\lambda \in [\lambda_i, \lambda_{i+1}]$. From the operator $D_{\eta_t}(\Theta) = \frac{\partial}{\partial t} + \delta_t(\Theta)$ and the well-known facts in [3, 5, 9], we have

$$\mathrm{Ind}D_{\eta_t}(\Theta) = \sum_{i=0}^{n} n_i.$$

This shows that $\mathrm{Ind}D_{\eta_t}(\Theta)$ is independent of the construction η_t and that is continuous in η_t. On the other hand,

$$\mathrm{Ind}D_{\eta_t}(\Theta) = \mu_{\eta_0}(\Theta) - \mu_{\eta_1}(\Theta).$$

Thus the obstruction to connect two generic perturbations is the spectral flow along the metric path in Σ_Y. The Riemannian-metric space Σ_Y is path-connected. So $\mathrm{Ind}D_{\eta_t}(\Theta) = 0$ provides that η_0 and η_1 are in the same (homotopy) class of with respect to the spectral flow. $\square$

Thus the dependence of metrics also enters into the definition of relative indices for $(a, \phi, \psi) \in \mathcal{R}^*_{SW}(Y, \eta)$. Now we follow the instanton case to fix the relative index

$$\mu_\eta(a, \phi, \psi) = \mathrm{Ind}D_\eta(\Theta, (a, \phi, \psi)) \in \mathbf{Z},$$

which depends on the value $\mu_\eta(\Theta)$. Any changes of $\mu_\eta(\Theta)$ shift $\mu_\eta(a, \phi, \psi)$ by an integer, and $\mu_\eta(\Theta)$ is understood with respect to some reference perturbation $\eta_0 \in \mathcal{P}_Y$.

Lemma 5.3. *For an admissible perturbation $\eta \in \mathcal{P}_Y$, the Seiberg-Witten moduli space $\mathcal{R}_{SW}(Y, \eta) = f_\eta^{-1}(0)$ is a compact 0-dimensional oriented manifold.*

Proof: The compactness can be proved by the 3-dimensional Weitzenböck formula and Mosers' weak maximal principle as in the 4-dimensional case [12, 26]. By the construction in the proof of Proposition 5.1, we can show that $\mathcal{R}_{SW}(Y, \eta) = f_\eta^{-1}(0)$ is a closed subset of the compact moduli space $\mathcal{M}_{Y \times S^1}(g + d\theta, \pi_1^* \eta)$, where $Y \times S^1$ carries the product metric $g + d\theta$. That $\mathcal{R}_{SW}(Y, \eta)$ is compact follows by Lemma 2 of [12]. By Proposition 4.6, $\mathcal{R}_{SW}(Y, \eta)$ is a 0-dimensional manifold. The orientation at each point of $\mathcal{R}_{SW}(Y, \eta)$ is defined by its spectral flow which depends on the perturbation homotopy class of η. (This is different phenomenon from the (instanton) Casson invariant of integral homology 3-spheres.) $\square$

Note that the monopole number $\#\mathcal{R}^*_{SW}(Y, \eta)$ (counted with sign) is not a topological invariant. The number $\#\mathcal{R}^*_{SW}(Y, \eta)$ depends on the metric with harmonic-spinor jumps.

6. Monopole homology of integral homology 3-spheres

For an admissible perturbation $\eta \in \mathcal{P}_Y$, we obtain a new gradient vector field f_η for which the irreducibles are all nondegenerate in §4. Since zeros of f_η are now isolated finite-many points, we use them to generate the monopole chain groups.

Definition 6.1. Let (a, ϕ, ψ) and (a', ϕ', ψ') be zeros of f_η. A *chain solution* $((A_1, \Phi_1), ..., (A_n, \Phi_n))$ from (a, ϕ, ψ) to (a', ϕ', ψ') is a finite set of Seiberg-Witten solutions over $Y \times \mathbf{R}$ which converge to $c_{i-1}, c_i \in f_\eta^{-1}(0)$ as $t \to \mp\infty$ such that $(a, \phi, \psi) = c_0$, $c_n = (a', \phi', \psi')$, and $(A_i, \Phi_i) \in \mathcal{M}_{Y \times \mathbf{R}}(c_{i-1}, c_i)$ for $0 \le i \le n$.

We say that the sequence $\{(A_\alpha, \Phi_\alpha)\} \in \mathcal{M}_{Y \times \mathbf{R}}((a, \phi, \psi), (a', \phi', \psi'))$ is *(weakly) convergent* to the chain solution $((A_1, \Phi_1), ..., (A_n, \Phi_n))$ if there is a sequence of n-tuples of real numbers $\{t_{\alpha,1} \le \cdots \le t_{\alpha,n}\}_\alpha$, such that $t_{\alpha,i} - t_{\alpha,i-1} \to \infty$ as $\alpha \to \infty$, and if, for each i, the translates $t_{\alpha,i}^*(A_\alpha, \Phi_\alpha) = (A_\alpha(\circ - t_{\alpha,i}), \Phi_\alpha(\circ - t_{\alpha,i}))$ converge weakly to (A_i, Φ_i).

Theorem 6.2. *Let* $\{(A_\alpha, \Phi_\alpha)\} \in \mathcal{M}_{Y \times \mathbf{R}}((a, \phi, \psi), (a', \phi', \psi'))$ *be a sequence of Seiberg-Witten solutions with uniformly bounded action over* $Y \times \mathbf{R}$. *Then there exists a subsequence converging to a chain solution* $((A_1, \Phi_1), ..., (A_n, \Phi_n))$ *such that*

$$IndD_{A_\alpha, \Phi_\alpha} = \sum_{i=1}^{n} IndD_{A_i, \Phi_i} = \sum_{i=1}^{n}(\mu_\eta(c_i) - \mu_\eta(c_{i-1})).$$

Proof: It follows from the same proof as in [9] §3 and [12], and the compactness of Seiberg-Witten moduli space on 4-dimensional manifolds. □

Proposition 6.3. *The compactification of* $\mathcal{M}_{Y \times \mathbf{R}}(c_0, c_{n+1})$ *with only chain solutions can be described as*

$$\overline{\mathcal{M}_{Y \times \mathbf{R}}(c_0, c_{n+1})} = \cup(\times_{i=1}^{n+1}\mathcal{M}_{Y \times \mathbf{R}}(c_{i-1}, c_i)),$$

the union over all sequence $c_0, c_1, \cdots, c_{n+1} \in \mathcal{R}_{SW}^*(Y, \eta)$ *such that* $\mathcal{M}_{Y \times \mathbf{R}}(c_{i-1}, c_i)$ *is nonempty for all* $1 \le i \le n+1$.

For any sequence $c_0, c_1, \cdots, c_{n+1} \in \mathcal{R}_{SW}^*(Y, \eta)$, *there is a gluing map*

$$G : \times_{i=1}^{n+1}\hat{\mathcal{M}}_{Y \times \mathbf{R}}(c_{i-1}, c_i) \times \Delta^{n+1} \to \overline{\mathcal{M}_{Y \times \mathbf{R}}(c_0, c_{n+1})},$$

where $\Delta^{n+1} = \{(\lambda_0, \cdots, \lambda_n) \in [-\infty, \infty]^{n+1} : 1 + \lambda_{i-1} < \lambda_i, 1 \le i \le n\}$.

1. *The image of G is a neighborhood of $\times_{i=1}^{n+1}\hat{\mathcal{M}}_{Y \times \mathbf{R}}(c_{i-1}, c_i)$ in the compactification with chain solutions.*
2. *The restriction of G to $\times_{i=1}^{n+1}\hat{\mathcal{M}}_{Y \times \mathbf{R}}(c_{i-1}, c_i) \times Int(\Delta^{n+1})$ is an orientation-preserving diffeomorphism onto its image.*

Proof: Since there is no bubbling in the Seiberg-Witten moduli space, the map G is the well-known transitivity in finite-dimensional Morse-Smale theory. □

Let $\mathcal{R}^n_{SW}(Y, \eta)$ be the set of irreducible zeros (a, ϕ, ψ) of f_η whose relative index $\mu_\eta(a, \phi, \psi) - \mu_\eta(\Theta) = n$. The **monopole chain group** $MC_n(Y, \eta)$ is defined to be the free Abelian group generated by $\mathcal{R}^n_{SW}(Y, \eta)$, where the admissible perturbation η specifies the spectral flow $\mu_\eta(\Theta)$. We write $I_\eta(\Theta; \eta_0)$ to be the integer $\mu_\eta(\Theta) - \mu_{\eta_0}(\Theta)$ with respect to a reference $\eta_0 \in \mathcal{P}_Y$. Hence $\mu_\eta(\Theta)$ is fixed with the fixation of $I_\eta(\Theta; \eta_0)$. Define the boundary operator $\partial : MC_n(Y, \eta) \to MC_{n-1}(Y, \eta)$:

$$\partial(a, \phi, \psi) = \sum_{(a', \phi', \psi') \in MC_{n-1}(Y, \eta)} \#\hat{\mathcal{M}}^1_{SW, Y \times \mathbf{R}}((a, \phi, \psi), (a', \phi', \psi')) \cdot (a', \phi', \psi').$$

Proposition 6.4. *Let* $\partial : MC_n(Y, \eta) \to MC_{n-1}(Y, \eta)$ *be defined as above. Then* $\partial \circ \partial = 0$.

Proof: The proof follows the same argument as in ([9], Theorem 2) except that we have to rule out the possibility of reducible connections entering into the picture. Note that

$$\partial^2(c_0) = \sum_{c_1 \in \mathcal{R}^{n-1}_{SW}(Y, \eta)} \sum_{c_2 \in \mathcal{R}^{n-2}_{SW}(Y, \eta)} \#\hat{\mathcal{M}}^1_{Y \times \mathbf{R}}(c_0, c_1) \cdot \#\hat{\mathcal{M}}^1_{Y \times \mathbf{R}}(c_1, c_2) c_2,$$

where $c_i = (a_i, \phi_i, \psi_i) \in \mathcal{R}^*_{SW}(Y, \eta) (i = 0, 1, 2)$. Consider in this sum all the terms associated to a fixed $c_2 \in \mathcal{R}^{n-2}_{SW}(Y, \eta)$. For the pair (c_0, c_2), there is the 2-dimensional moduli space $\mathcal{M}^2_{Y \times \mathbf{R}}(c_0, c_2)$. By Proposition 6.3, the ends of $\hat{\mathcal{M}}^2_{Y \times \mathbf{R}}(c_0, c_2)$ consists of all the components $\hat{\mathcal{M}}^1_{Y \times \mathbf{R}}(c_0, c_1) \times \hat{\mathcal{M}}^1_{Y \times \mathbf{R}}(c_1, c_2)$ with $c_1 \in \mathcal{R}^{n-1}_{SW}(Y, \eta)$. It is impossible for c_1 to be the $U(1)$-reducible zero of f_η because the isotropy subgroup Γ_{c_1} would add to the gluing parameter and as a result would contradict the dimension count by Proposition 5.1 and Proposition 5.2. Thus

$$\sum_{c_1 \in \mathcal{R}^{n-1}_{SW}(Y, \eta)} \#\hat{\mathcal{M}}^1_{Y \times \mathbf{R}}(c_0, c_1) \cdot \#\hat{\mathcal{M}}^1_{Y \times \mathbf{R}}(c_1, c_2) = \partial\hat{\mathcal{M}}^2_{Y \times \mathbf{R}}(c_0, c_2) = 0.$$

$\square$

As a consequence of Proposition 6.4, for a given integral homology 3-sphere Y and an admissible data $\eta \in \mathcal{P}_Y$, we have a well-defined definition of a **Monopole Homology**

$$MH_*(Y; \eta) = \ker \partial_* / \mathrm{Im}\partial_{*+1}, \quad * \in \mathbf{Z}.$$

Now the monopole homology $MH_*(Y; \eta)$ is sensitive to the number $I_\eta(\Theta; \eta_0)$, and $MH_*(Y; \eta)$ is not a topological invariant since its Euler characteristic $\#\mathcal{R}^*_{SW}(Y, \eta)$ is metric-dependent.

7. HOMOMORPHISMS INDUCED BY COBORDISMS

From the troublesome path of metrics in Σ_Y of creating/destroying harmonic spinors (see [11]), the invariance of the monopole homology of integral homology 3-spheres is in question. The cobordism argument used in [9] does not apply here. We have to construct a different cobordism between metrics and admissible perturbations with the fixed spectral flow $I_\eta(\Theta; \eta_0) = \mu_\eta(\Theta) - \mu_{\eta_0}(\Theta)$. In this section, we show that our monopole homology is independent of metrics and of admissible perturbations within the class $I_\eta(\Theta; \eta_0)$.

Let X be an oriented 4-manifold with two cylindrical ends $Y_1 \times \mathbf{R}_+$ and $Y_2 \times \mathbf{R}_-$, where Y_1 and Y_2 are integral homology 3-spheres. Let $\tau : X \to [0, \infty)$ be a smooth cutoff function such that

$\tau(x) = 0$ for x lying outside of $Y_1 \times \mathbf{R}_+ \cup Y_2 \times \mathbf{R}_-$ and $\tau(y,t) = |t|$ for $(y,t) \in Y_1 \times \mathbf{R}_+ \cup Y_2 \times \mathbf{R}_-$ and $|t| > t_0 > 0$ and $e_\delta = e^{\delta \tau(x)}$. Then using the cutoff function τ and a background connection we can extend $\frac{d}{dt} + \alpha, \frac{d}{dt} + \beta$ to a connection ∇_0 on X such that

$$\nabla_0|_{Y_1 \times [t_0, \infty)} = \frac{d}{dt} + \alpha, \quad \nabla_0|_{Y_2 \times (-\infty, -t_0]} = \frac{d}{dt} + \beta.$$

Similarly, we can extend sections on $W_X^\pm$. The Fréchet space $\Omega^1_{\text{comp}}(X, AdP) \oplus \Gamma_{\text{comp}}(W_X^\pm)$ of compact supported C^∞-sections on $(T^*X \otimes AdP) \oplus \Gamma(W_X^\pm)$ can be completed to a Banach space

$$\mathcal{A}^p_{k,\delta}(X) = (\nabla_0, 0) + L^p_{k,\delta}(\Omega^1(X, AdP) \oplus \Gamma(W_X^\pm)),$$

where $\|c\|_{L^p_{k,\delta}} = \|e_\delta \cdot c\|_{L^p_k}$ for $c \in \Omega^1_{\text{comp}}(X, AdP) \oplus \Gamma_{\text{comp}}(W_X^\pm)$. The gauge group $\mathcal{G}^p_{k+1,\delta}$ is given by $L^p_{k+1,\delta}$-norm of $\text{Aut}(\det W_X^\pm)$. So the quotient space is $\mathcal{B}^p_{k,\delta}(X) = \mathcal{A}^p_{k,\delta}(X)/\mathcal{G}^p_{k+1,\delta}$. The perturbation data $\eta_1 = (g_{Y_1}, \alpha_1)$ and $\eta_2 = (g_{Y_2}, \alpha_2)$ at the ends provide the gradient vector fields f_{η_1} and f_{η_2} so that the zeros of f_{η_1} on Y_1 and of f_{η_2} on Y_2 are generic. Clearly these perturbation data η_1 and η_2 can be pulled back to the cylindrical ends $Y_1 \times \mathbf{R}_+$ and $Y_2 \times \mathbf{R}_-$, and produce perturbations on the time-invariant monopole equation on $\mathcal{B}^p_{k,\delta}(Y_1 \times \mathbf{R}_+)$ and $\mathcal{B}^p_{k,\delta}(Y_2 \times \mathbf{R}_-)$ (same δ as before). According to ([9] (1c.2) and [12, 21, 26]), there exists a Baire's first category subset in the space $\mathcal{M}et(X) \times \Pi_X$ of Riemannian metrics g_X and perturbation data α_X such that $\mathcal{M}_{\eta_X}(c, c')$ $(\eta_X = (g_X, \alpha_X))$ is a smooth manifold with

$$\dim \mathcal{M}_{\eta_X}(c, c') = \mu_{\eta_1}(c) - \mu_{\eta_2}(c') + \frac{1}{2}(2\chi + 3\sigma)(X). \tag{7.1}$$

In addition, $\mathcal{M}_{\eta_X}(c, c')$ is oriented with an orientation specified by the orientations on $H^1(X, \mathbf{R})$ and $H^0(X, \mathbf{R}) \oplus H^2_+(X, \mathbf{R})$ (see [6, 12, 21, 26]).

Define a homomorphism $\Psi_* = \Psi_*(X; \eta_X) : MC_*(Y_1; \eta_1) \to MC_*(Y_2; \eta_2)$ of the monopole chain complexes by the formula

$$\Psi_*(c) = \sum_{c' \in \mathcal{R}^*_{SW}(Y_2, \eta_2)} \#\mathcal{M}^0_{\eta_X}(c, c') \cdot c', \quad c \in \mathcal{R}^*_{SW}(Y_1, \eta_1),$$

where $\mathcal{M}^0_{\eta_X}(c, c')$ is the 0-dimensional oriented moduli space connecting c to c' on X and $\mu_{\eta_1}(c) - \mu_{\eta_2}(c') = -\frac{1}{2}(2\chi + 3\sigma)(X)$.

Proposition 7.1. *Given a cobordism X and perturbation data $\eta_X \in \mathcal{M}et(X) \times \Pi_X$ as before, the homomorphism Ψ_* is a chain map shifting the degree by $\frac{1}{2}(2\chi + 3\sigma)(X)$. Furthermore the induced homomorphism*

$$\Psi_* = \Psi_*(X; \eta_X) : MH_*(Y_1; \eta_1) \to MH_*(Y_2; \eta_2)$$

on the monopole homologies depends only on the cobordism X.

Proof: It follows the same argument as in [9] Theorem 3 and [13] §5.   □

We show below that $\Psi_*(X; \eta_X)$ is functorial with respect to the composite cobordism. Given two cobordisms $(U; \eta_U)$ connecting Y_1 to Y_2 and $(V; \eta_V)$ connecting Y_2 to Y_3 so that η_U and η_V agree

on Y_2, we can form the composite cobordism $(W; \eta_W)$ connecting Y_1 to Y_3. Then

$$\Psi_*(W; \eta_W) = \Psi_*(V; \eta_V) \circ \Psi_*(U; \eta_U). \tag{7.2}$$

A different strategy from Floer's has to be taken to prove that $MH_*(Y, \eta)$ is independent of admissible perturbations $\eta = (g_Y, \alpha)$ within the class of $I_\eta(\Theta; \eta_0)$. We consider the time-dependent perturbations of the Seiberg-Witten equation and its associated moduli space. Given two admissible perturbation data of generic metrics g_Y^{-1} and g_Y^1 and 1-forms α_{-1} and α_1 with $I_{\eta_{-1}}(\Theta; \eta_0) = I_{\eta_1}(\Theta; \eta_0)$ (here $\eta_t = (g_Y^t, \alpha_t)$), there is an one-parameter family of admissible perturbations $\Lambda = \{\eta_t = (g_Y^t, \alpha_t)| -\infty \leq t \leq \infty\}$ joining them. Assume that the pair $\eta_t = (g_Y^{-1}, \alpha_{-1})$ for $t \leq -1$ and $\eta_t = (g_Y^1, \alpha_1)$ for $t \geq 1$. On the cylinder $Y \times \mathbf{R}$, we consider the perturbed Seiberg-Witten equation

$$\frac{\partial \psi}{\partial t} + \partial_{a_t}^{\nabla^{g_Y^t} + \alpha_t} \psi = 0, \quad \frac{\partial a_t}{\partial t} + *_{g_Y^t} F(a_t) + \alpha_t = i\tau_{g_Y^t}(\psi, \psi). \tag{7.3}$$

Given $c \in \mathcal{R}_{SW}^*(Y, \eta_{-1})$ and $c' \in \mathcal{R}_{SW}^*(Y, \eta_1)$, we denote by $\mathcal{M}_\Lambda(c, c')$ the subspace in $\mathcal{B}_{k,\delta}^p(c, c')$ consisting of solutions of (7.3). Then there exists a homomorphism

$$\Psi_\Lambda : MC_n(Y; \eta_{-1}) \to MC_n(Y; \eta_1)$$

of the monopole chain complexes defined by

$$\Psi_\Lambda(c) = \sum_{c' \in \mathcal{R}_{SW}^n(Y, \eta_1)} \#\mathcal{M}_\Lambda^0(c, c') \cdot c', \quad c \in \mathcal{R}_{SW}^n(Y, \eta_{-1}).$$

Proposition 7.2. *Let $\Lambda = \{\eta_t = (g_Y^t, \alpha_t)| t \in \mathbf{R}\}$ be an family of admissible perturbations as defined above such that $Ind D_{\eta_t}(\Theta) = 0$. Then*

1. *If Λ is a constant family of admissible perturbations $(g_Y^t = g_Y, \alpha_t = \alpha)$, then $\Psi_\Lambda = id$.*
2. *Ψ_Λ is a chain map: $\partial \Psi_\Lambda = \Psi_\Lambda \partial$.*
3. *Given two families Λ and Λ' of admissible perturbations joining (g_Y^{-1}, α_{-1}) to (g_Y^0, α_0) and from (g_Y^0, α_0) to (g_Y^1, α_1), we have $\Psi_{\Lambda \circ \Lambda'} = \Psi_\Lambda \circ \Psi_{\Lambda'}$.*
4. *If a family Λ_0 of admissible perturbations connecting (g_Y^{-1}, α_{-1}) and (g_Y^1, α_1) can be deformed into another Λ_1 by admissible families $\Lambda_\lambda (0 \leq \lambda \leq 1)$, then the two monopole chain maps Ψ_{Λ_0} and Ψ_{Λ_1} are chain homotopic to each other.*

Proof: (1) If the perturbation is time independent $\eta_t = (g_Y, \alpha)$, then $\mathcal{M}_\Lambda^0(c, c')$ is just the space $\mathcal{M}_{Y \times \mathbf{R}}^0(c, c')$. For the 0-dimensional component $\mathcal{M}_\Lambda^0(c, c')$, this means time-invariant solutions c_t on $Y \times \mathbf{R}$, and we have $[c_t] = c = c'$. Therefore $\#\mathcal{M}_\Lambda^0(c, c') = \delta_{cc'}$ and $\Psi_\Lambda = id$.

(2) We consider the compactification of $\mathcal{M}_\Lambda(c, c')$ as developed in [10, 13]. By Proposition 6.3 and [12, 21, 26], $\mathcal{M}_\Lambda(\alpha, \beta)$ can be compactified such that the codimension-one boundary consists of

$$\cup_{c_{-1}} \hat{\mathcal{M}}_{Y \times \mathbf{R}}(c, c_{-1}) \times_{c_{-1}} \mathcal{M}_\Lambda(c_{-1}, c') \coprod \cup_{c_1} \mathcal{M}_\Lambda(c, c_1) \times_{c_1} \hat{\mathcal{M}}_{Y \times \mathbf{R}}(c_1, c'). \tag{7.4}$$

Here $c_{\pm 1} \in \mathcal{R}_{SW}(Y, \eta_{\pm 1})$ and $\mathcal{M}_{Y \times \mathbf{R}}(c, c_{-1})$ is the moduli space of monopoles on $Y \times (-\infty, -1)$ with respect to the perturbation η_{-1} and $\hat{\mathcal{M}}_{Y \times \mathbf{R}}(c, c_{-1}) = \mathcal{M}_{Y \times \mathbf{R}}(c, c_{-1})/\mathbf{R}$. Similarly $\hat{\mathcal{M}}_{Y \times \mathbf{R}}(c_1, c')$ is obtained from the perturbation data η_1. Consider the 1-dimensional components $\mathcal{M}_\Lambda^1(c, c')$ of

$\mathcal{M}_\Lambda(c, c^{'})$, whose boundary by (7.4) gives two types of oriented points counted as $\partial \Psi_\Lambda = \Psi_\Lambda \partial$. We can rule out the possibilities of the reducible Θ for $c_{\pm 1}$. If they occurred, then they would have an additional $U(1)$-symmetry on these moduli spaces. This is impossible by the dimension reasoning from Proposition 5.1, Proposition 5.2 and our hypothesis $I_{\eta_{-1}}(\Theta; \eta_0) = I_{\eta_1}(\Theta; \eta_0)$ (see below also).

(3) For a composite cobordism and its induced homomorphism, we study the moduli space $\mathcal{M}_{\Lambda * \Lambda^{'}}(T; \alpha, \beta)$ of solutions of the Seiberg-Witten equation on $Y \times \mathbf{R}$ with respect to the following time-dependent admissible perturbation data $\Lambda *_T \Lambda^{'}$, where

$$\Lambda *_T \Lambda^{'} = \begin{cases} \eta_{-1} = (g_Y^{-1}, \alpha_{-1}) & -\infty < t \le -T - 1 \\ \Lambda = (g_Y^{t+T}, \alpha_{t+T}) & -T - 1 \le t \le -T \\ \eta_0 & -T \le t \le T \\ \Lambda^{'} = (g_Y^{t-T}, \alpha_{t-T}) & T \le t \le T + 1 \\ \eta_1 & T + 1 \le t < +\infty. \end{cases}$$

Let T be sufficiently large. Thus $\mathcal{M}_{\Lambda * \Lambda^{'}}(T; c, c^{'})(T \ge T_0)$ is approximated by the union

$$\cup_{c_0} \overline{\mathcal{M}}_\Lambda(c, c_0) \times_{c_0} \overline{\mathcal{M}}_{\Lambda^{'}}(c_0, c^{'}). \tag{7.5}$$

where $\overline{\mathcal{M}}_\Lambda(c, c_0) = \mathcal{M}_\Lambda(c, c_0)/(\Gamma_c \times \Gamma_{c_0})$. Note that the 0-dimensional components in $\overline{\mathcal{M}}_\Lambda(c, c_0) \times_{c_0} \overline{\mathcal{M}}_{\Lambda^{'}}(c_0, c^{'})$ correspond to the $c^{'}$-coefficients in

$$\Psi_{\Lambda^{'}} \circ \Psi_\Lambda(c) = \sum_{c_0} \#\overline{\mathcal{M}}_\Lambda^0(c, c_0) \cdot \#\overline{\mathcal{M}}_{\Lambda^{'}}^0(c_0, c^{'}) \cdot c^{'}.$$

On the other hand, as $T \to 0$, the 0-dimensional component of the moduli space $\mathcal{M}_{\Lambda * \Lambda^{'}}(T; c, c^{'})$ gives the $c^{'}$-coefficients in $\Psi_{\Lambda * \Lambda^{'}}(c) = \sum \mathcal{M}_{\Lambda * \Lambda^{'}}^0(c, c^{'}) \cdot c^{'}$. Because $\cup_{0 \le T \le T_0} \mathcal{M}_{\Lambda * \Lambda^{'}}^0(T; c, c^{'})$ is the cobordism between $\mathcal{M}_{\Lambda * \Lambda^{'}}^0(0; c, c^{'})$ and $\mathcal{M}_{\Lambda * \Lambda^{'}}^0(T_0; c, c^{'})$, so the assertion (3) follows by ruling out the reducible Θ. Note that

$$\dim \overline{\mathcal{M}}_\Lambda(c, c_0) = \mu_{\eta_{-1}}(c) - \lim_{\eta_t \in \Lambda, \eta_t \to \eta_0} \mu_{\eta_t}(c_0) - \dim \Gamma_{c_0};$$

$$\dim \overline{\mathcal{M}}_{\Lambda^{'}}(c_0, c^{'}) = \lim_{\eta_t \in \Lambda^{'}, \eta_t \to \eta_0} \mu_{\eta_t}(c_0) - \mu_{\eta_1}(c^{'}). \tag{7.6}$$

By Proposition 5.1 and Proposition 5.2, we obtain

$$\lim_{\eta_t \in \Lambda, \eta_t \to \eta_0} \mu_{\eta_t}(c_0) = \lim_{\eta_t \in \Lambda^{'}, \eta_t \to \eta_0} \mu_{\eta_t}(c_0) = \mu(c_0).$$

So it satisfies the equations $\mu_{\eta_{-1}}(c) - \mu(c_0) = 1$ $(c_0 = \Theta)$ and $\mu(c_0) - \mu_{\eta_1}(c^{'}) = 0$. This is impossible because of $\mu_{\eta_{-1}}(c) = \mu_{\eta_1}(c^{'})$. *If these spectral flows $I_{\eta_{\pm 1}}(\Theta; \eta_0)$ are not fixed to be same, then the above argument becomes invalid.*

(4) Let $\Lambda_i(i = 0, 1)$ be a family of time-independent admissible perturbations which connect up η_{-1} and η_1. Suppose that Λ_0 and Λ_1 can be smoothly deformed from one to another by a 1-parameter family $\Lambda_s = \{\eta_t^s = (g_Y^{s;t}, \alpha_t^s), 0 \le s \le 1, -1 \le t \le 1\}$ of the same type of admissible perturbations. Set $\Lambda_s = \Lambda_0$ for $0 \le s \le \frac{1}{4}$ and $\Lambda_s = \Lambda_1$ for $\frac{3}{4} \le s \le 1$. Associated to this situation, there is a 1-parameter family of moduli spaces denoted by $\mathcal{H}\tilde{\mathcal{M}}(c, c^{'}) = \cup_{0 \le s \le 1} \tilde{\mathcal{M}}_{\Lambda_s}(c, c^{'})$,

$$\mathcal{H}\tilde{\mathcal{M}}(c, c^{'}) = \{(\Phi, s) | \Phi \in \tilde{\mathcal{M}}_{\Lambda_s}(c, c^{'}), 0 \le s \le 1\} \subset \mathcal{B}_{k, \delta}^p(c, c^{'}) \times [0, 1],$$

where $\mathcal{H}\tilde{\mathcal{M}}$ is the set of regular solutions of Seiberg-Witten equation with respect to η_t^s, and is a smooth manifold with dimension $\mu_{\eta_{-1}}(c) - \mu_{\eta_1}(c^{'}) + 1$. The codimension-one boundary consists of

$$\mathcal{M}_{\Lambda_1}(c, c^{'}) \times \{0\} \coprod \mathcal{M}_{\Lambda_0}(c, c^{'}) \times \{1\},$$

$$\cup_{(s,c_0)}\tilde{\mathcal{M}}_{\Lambda_s}(c, c_0) \times \mathcal{M}_{\eta_1}(c_0, c^{'}) \coprod \cup_{(s,\gamma)}\mathcal{M}_{\eta_{-1}}(c, c_0) \times \tilde{\mathcal{M}}_{\Lambda_s}(c_0, c^{'}).$$

Since $\tilde{\mathcal{M}}_{\Lambda_s}(c, c_0)$ and $\tilde{\mathcal{M}}_{\Lambda_s}(c_0, c^{'})$ are solutions of the Seiberg-Witten equation with virtual dimension -1, they can only occur for $0 < s < 1$. The homomorphism $H : MC_*(Y; \eta_{-1}) \rightarrow MC_*(Y; \eta_1)$ of degree $+1$ is defined by

$$H(c) = \sum_{c_0}\sum_s \#\tilde{\mathcal{M}}_{\Lambda_s}^0(c, c_0) \cdot c_0, \quad \text{for} \ \ c \in \mathcal{R}_{SW}^n(Y, \eta_{-1}), c_0 \in \mathcal{R}_{SW}^{n+1}(Y, \eta_1).$$

That c_0 is reducible is eliminated by the extra $U(1)$-symmetries in $\mathcal{M}_{\eta_1}(c_0, c^{'})$ and $\mathcal{M}_{\eta_{-1}}(c, c_0)$ and $I_{\eta_1}(\Theta; \eta_0) = I_{\eta_{-1}}(\Theta; \eta_0)$. Summing up $c^{'} \in \mathcal{R}_{SW}^n(Y, \eta_1)$, we have

$$\Psi_{\Lambda_0}(c) - \Psi_{\Lambda_1}(c) = H \circ \partial_{\eta_{-1}}(c) + \partial_{\eta_1} \circ H(c).$$

Therefore Ψ_{Λ_0} and Ψ_{Λ_1} are monopole chain homotopic to each other. $\qquad\square$

Thus the monopole homology groups $MH_*(Y; \eta^{\pm 1})$ associated to two admissible perturbation data are canonically isomorphic to each other whenever $I_{\eta^1}(\Theta; \eta_0) = I_{\eta^{-1}}(\Theta; \eta_0)$ for the unique $U(1)$-reducible Θ on Y. Thus it is more appropriate to denote $MH_*(Y; \eta)$ by $MH_*(Y; I_\eta(\Theta; \eta_0))$. For an integral homology 3-sphere Y, the monopole homology can be extended to a function

$$MH_{SWF} : \{I_\eta(\Theta; \eta_0) : \eta \in \mathcal{P}_Y\} \rightarrow \{MH_*(Y, I_\eta(\Theta; \eta_0)) : \eta \in \mathcal{P}_Y\}.$$

(Changing a reference η_0 corresponds to the same homology groups with grading $I_{\eta_0'}(\Theta; \eta_0)$-shift) This function MH_{SWF} is a topological invariant of the integral homology 3-sphere Y, up to the degree-shifting of monopole homologies. Hence such a function MH_{SWF} may be called a Seiberg-Witten-Floer theory, which is completely different from the instanton Floer homology, but more related to the treatment in [13].

8. Relative Seiberg-Witten invariants

The Seiberg-Witten invariant (see [6, 21, 26]) has proved so useful and at least powerful as the Donaldson invariant in many cases, and is much easier to compute. In this section we are going to extend the Seiberg-Witten invariant to the relative one on smooth 4-manifolds with boundary integral homology 3-spheres. The "relative Seiberg-Witten invariants" is no longer a topological invariant since it lies in a monopole homology depending upon Riemannian metrics of integral homology 3-spheres. But the natural pairing between "relative Seiberg-Witten invariants" does recover the Seiberg-Witten invariant of closed smooth 4-manifolds.

Let X be a smooth 4-manifold with $b_1(X) > 0$ and boundary Y (an integral homology 3-sphere). The collar of X can be identified with $Y \times [-1, 1]$, and the admissible perturbation data on Y can be extended inside X as we did in §7. Fixing $I_\eta(\Theta; \eta_0)$ should be understood though this section.

Definition 8.1. For a smooth 4-manifold X with boundary Y (an integral homology 3-sphere), the 0-degree relative Seiberg-Witten invariant is defined by

$$q_{X,Y,\eta} = \sum_{c \in \mathcal{R}^*_{SW}(Y,\eta)} \#\mathcal{M}^0_X(c) \cdot c,$$

where $\mathcal{R}^*_{SW}(Y,\eta)$ is the set of all nondegenerate zeros of f_η with prescribed $I_\eta(\Theta;\eta_0)$.

By the index calculation and our convention $\mu_\eta(c) = SF(c,\Theta)$, we have

$$\dim\mathcal{M}^0_X(c) + \mu_\eta(c) = \dim\mathcal{M}_X(\Theta) = \frac{1}{4}(c_1(\pi^*(L))^2 - (2\chi + 3\sigma))(X) = -\frac{1}{4}(2\chi + 3\sigma)(X),$$

since $c_1(L) = 0$ for the integral homology 3-sphere Y. Thus $q_{X,Y,\eta}$ is in the monopole chain group with grading $-\frac{1}{4}(2\chi + 3\sigma)(X)$.

Proposition 8.2. *For $q_{X,Y,\eta} \in MC_{\mu_X}(Y,\eta)$ with $\mu_X = -\frac{1}{4}(2\chi + 3\sigma)(X)$ and a fixed class $I_\eta(\Theta;\eta_0)$, we have $\partial_Y \circ q_{X,Y,\eta} = 0$.*

Proof:

$$\partial_Y \circ q_{X,Y,\eta}(c) = \sum_{c \in \mathcal{R}^\mu_{SW}(Y,\eta)} \sum_{c' \in \mathcal{R}^{\mu-1}_{SW}(Y,\eta)} \#\mathcal{M}^0_X(c) \cdot \#\hat{\mathcal{M}}^1_{Y \times \mathbf{R}}(c,c') \cdot c'.$$

For both c and c' irreducible (nondegenerate) zeros of f_η, we take one-dimensional moduli space $\mathcal{M}^1_X(c')$ for fixed c'. Then we count the ends of the moduli space to conclude the result. Again it is a technical point to avoid the reducible Θ entering the boundary $\mathcal{M}_X(\Theta) \times \mathcal{M}_{Y \times \mathbf{R}}(\Theta,c')$. For the reducible Θ, we have the dimension counting

$$\dim\{\mathcal{M}_X(\Theta) \times \mathcal{M}_{Y \times \mathbf{R}}(\Theta,c')\} = \dim\mathcal{M}_X(\Theta) + \dim\Gamma_\Theta + \dim\mathcal{M}_{Y \times \mathbf{R}}(\Theta,c') \geq 0 + 1 + 1 = 2.$$

So c cannot be the reducible Θ, and $\partial_Y \circ q_{X,Y,\eta} = 0$. Hence $q_{X,Y,\eta}$ is indeed a monopole cycle. $\qquad\square$

Let $q_{X,Y,\eta}(g_X)$ be the relative Seiberg-Witten invariant with respect to the metric g_X. Now we show that the monopole homology class $[q_{X,Y,\eta}(g_X)]$ defined by Proposition 8.2 is independent of metrics g_X with $g_X|_Y$ in the fixed class of $I_\eta(\Theta;\eta_0)$.

Proposition 8.3. *Let $g^i_X (i = 1, 2)$ be two generic metrics on X with induced metric g^i_Y generic such that $I_{\eta_1}(\Theta;\eta_0) = I_{\eta_2}(\Theta;\eta_0)$ and $\eta_i = (g^i_Y, \alpha_i)$. Then there exist $c' \in MC_{\mu_X+1}$ with $\mu_X = -\frac{1}{4}(2\chi + 3\sigma)(X)$ such that we have*

$$q_{X,Y,\eta_2}(g^2_X) - q_{X,Y,\eta_1}(g^1_X) = \partial(c').$$

In particular, $[q_{X,Y,\eta_2}(g^2_X)] = [q_{X,Y,\eta_1}(g^1_X)]$ as the monopole homology class in $MH_{\mu_X}(Y, I_{\eta_i}(\Theta;\eta_0))$.

Proof: Let $\{g^{t+1}_X\}_{0 \leq t \leq 1}$ be a family of metrics on X such that $I_{\eta_{t+1}}(\Theta;\eta_0)$ is independent of t with $\eta_{t+1} = (g^{t+1}_X|_Y, \alpha_{t+1})$ and $\mathcal{M}^0_X(g^{t+1}_X)(c)$ has virtual dimension 0 with respect to c irreducible. Therefore $\{\mathcal{M}^0_X(g^{t+1}_X)(c)\}_{0 \leq t \leq 1}$ is an one-dimensional moduli space of Seiberg-Witten solutions on X. The corresponding codimension-one boundary in $[0,1] \times \mathcal{B}_X(g^{t+1}_X)(c)$ is given by

$$\partial(\{\mathcal{M}^0_X(g^{t+1}_X)(c)\}_{0 \leq t \leq 1}) =$$

$$\{0\} \times \mathcal{M}_X^0(g_X^1)(c) \coprod -\{1\} \times \mathcal{M}_X^0(g_X^2)(c) \coprod \partial(\sum_{\mu_{\eta_{t+1}}(c)-\mu_{\eta_{t+1}}(c')=-1} \#([0,1] \times \mathcal{M}_X^{-1}(g_X^{t+1})(c'))).$$

The number $\langle \partial_Y c', c \rangle$ is the algebraic number of $([0,1] \times \mathcal{M}_X^{-1}(g_X^{t+1})(c'))$. The c' cannot be the reducible Θ by the fixed $I_{\eta_1}(\Theta; \eta_0)$ with the same argument as before. So

$$q_{X,Y,\eta_2}(g_X^2)(c) - q_{X,Y,\eta_1}(g_X^1)(c) = \langle \partial_Y c', c \rangle.$$

Hence $q_{X,Y,\eta_i}(g_X^i)(i = 1, 2)$ (as a monopole cycle) gives the same monopole homology class. □

Note that orientation reversing from Y to $-Y$ changes the grading from $\mu_\eta(c)$ to $-1 - \mu_\eta(c)$ (certainly does not change the solutions of the Seiberg-Witten equation on the 3-manifold), so there is a nature identification between $MC_{\mu_\eta}(Y, \eta)$ and $CF_{-1-\mu_\eta}(-Y, \eta)$.

Theorem 8.4. *For a smooth 4-manifold $X = X_0 \#_Y X_1$ with $b_2^+(X_i) > 0 (i = 0, 1)$ and Y an integral homology 3-sphere, the Seiberg-Witten invariant of the 4-manifold X is given by the Kronecker pairing of $MH_*(Y; I_\eta(\Theta; \eta_0))$ with $MH_{-1-*}(-Y; I_\eta(\Theta; \eta_0))$ for $q_{X_0,Y,\eta}$ and $q_{X_1,-Y,\eta}$;*

$$\langle, \rangle : MH_*(Y; I_\eta(\Theta; \eta_0)) \times MH_{-1-*}(-Y; I_\eta(\Theta; \eta_0)) \to \mathbf{Z}; \quad q_{SW}(X) = \langle q_{X_0,Y,\eta}, q_{X_1,-Y,\eta} \rangle.$$

More precisely, $q_{SW}(X_0 \#_Y X_1) = \sum_c \#\mathcal{M}_{X_0,Y,\eta}^0(c) \cdot \#\mathcal{M}_{X_1,-Y}^0(-c)$, where $I_\eta(\Theta; \eta_0)$ is fixed. The invariant $q_{SW}(X)$ is independent of the choice of $I_\eta(\Theta; \eta_0)$.

Proof: If Y admits a metric of positive scalar curvature, then the proof is given in [26] with $I_\eta(\Theta; \eta_0) = 0$ the special case. The assumption implies that $b_2^+(X) > 1$. So we can rule out the existence of reducible solutions on X by the standard method (see [6, 12, 21, 26]). Note that

$$\dim\mathcal{M}_{X_0}(c) + \dim\mathcal{M}_{X_1}(c) + \dim\Gamma_\Theta = \dim\mathcal{M}_X.$$

By the dimension equation, we can eliminate the term $\#\mathcal{M}_{X_0,Y,\eta}^0(c) \cdot \#\mathcal{M}_{X_1,-Y,\eta}^0(-c)$ with $c = \Theta$. Then the 0-dimensional moduli space on X is obtained by gluing the solutions on (X_0, Y) with ones on $(X_1, -Y)$. Using the standard technique on stretching the neck [7], one gets the equality $q_{SW}(X) = \langle q_{X_0,Y,\eta}, q_{X_1,-Y,\eta} \rangle$. Since $q_{SW}(X)$ is a topological invariant, so the pairing is independent of the choice of $I_\eta(\Theta; \eta_0)$. □

For higher degree relative Seiberg-Witten invariants, one can obtain the similar results as in [13].

Computing the monopole homology is extremely complicated due to the Riemannian metric, harmonic spinor, spectral flow and solution of the first-order Dirac-type nonlinear differential equation. Even for the 3-sphere, a complete calculation of the function MH_{SWF} is very difficult at this moment. Understand the harmonic spinors on S^3 with a subfamily of Riemannian metrics (metrics are $SU(2)$-left invariant and $U(1)$-right invariant) is already quite involved by the work of Hitchin [11]. On the other hand, Theorem 8.4 gives us a flexibility to understand the Seiberg-Witten invariant of closed smooth 4-manifolds through the relative ones with some preferred Riemannian metric(s) on the integral homology 3-sphere.

Remark: The method we developed in this paper also can be extended to rational homology 3-spheres with fixed spectral flows along all $U(1)$-reducible solutions of Seiberg-Witten equation on the rational homology 3-sphere (see [13] for more detail).

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